Section 7.5: Using the Laplace Transform to Solve IVPs

Section 7.5: Using the Laplace Transform to Solve IVPs

Method of Laplace Transforms

To solve an initial value problem:

(a) Take the Laplace transform of both sides of the equation.
(b) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.
(c) Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

In step (a) we are tacitly assuming the solution is piecewise continuous on $[0, \infty)$ and of exponential order. Once we have obtained the inverse Laplace transform in step (c), we can verify that these tacit assumptions are satisfied.

Section 7.5: Using the Laplace Transform to Solve IVPs

Example 1 Solve the initial value problem

(1) $$y'' - 2y' + 5y = -8e^{-t}; \quad y(0) = 2, \quad y'(0) = 12.$$

Section 7.5: Using the Laplace Transform to Solve IVPs

Example 2 Solve the initial value problem

(4) $$y'' + 4y' - 5y = te^t\,; \qquad y(0) = 1\,, \qquad y'(0) = 0\,.$$

TABLE 7.1 **Brief Table of Laplace Transforms**

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2 + b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$

Section 7.5: Using the Laplace Transform to Solve IVPs

Example 3 Solve the initial value problem

(8) $$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}; \quad w(\pi) = 2, \quad w'(\pi) = 12.$$

Section 7.5: Using the Laplace Transform to Solve IVPs

Example 4 Solve the initial value problem

(13) $$y'' + 2ty' - 4y = 1\,, \qquad y(0) = y'(0) = 0\,.$$

Discussion

- When coefficients aren't constant, you end up getting a DE in Y(s)
- Often we need the result below

In illustrating the technique, we make use of the following fact. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, then*

(12) $$\lim_{s\to\infty} \mathscr{L}\{f\}(s) = 0\,.$$

Section 7.5: Using the Laplace Transform to Solve IVPs

Example 4 Solve the initial value problem

(13) $$y'' + 2ty' - 4y = 1\,, \qquad y(0) = y'(0) = 0\,.$$

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$